Supplementary Information

First-passage times in complex scale invariant media, by Condamin et al.

I. DERIVATION OF EQUATION (2) OF THE MAIN TEXT

We start from the following equation for the propagator W and the FPT density P:

$$W(\mathbf{r}_T, t|\mathbf{r}_S) = \int_0^t P(\mathbf{r}_T, t'|\mathbf{r}_S) W(\mathbf{r}_T, t-t'|\mathbf{r}_T) dt' \tag{1}$$

Integrating this equation, and inverting the integrals, we get:

$$\int_0^T W(\mathbf{r}_T, t|\mathbf{r}_S) dt = \int_0^T dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \int_{t'}^T dt W(\mathbf{r}_T, t'-t|\mathbf{r}_T) \tag{2}$$

We now define

$$H(\mathbf{r}|\mathbf{r}') = \int_0^\infty (W(\mathbf{r}, t|\mathbf{r}') - W_{\text{stat}}(\mathbf{r})) dt, \tag{3}$$

where W_{stat} is the stationary probability distribution. The integrals (2) can then be evaluated as a function of H and W_{stat}, up to a correction $\mu(T)$:

$$H(\mathbf{r}_T|\mathbf{r}_S) + TW_{\text{stat}}(\mathbf{r}_T) = \int_0^T P(\mathbf{r}_T, t'|\mathbf{r}_S)[H(\mathbf{r}_T|\mathbf{r}_T) + (T-t')W_{\text{stat}}(\mathbf{r}_T)]dt' + \mu(T), \tag{4}$$

where

$$\mu(T) = \int_0^T dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \left[\int_0^{T-t'} dt (W(\mathbf{r}_T, t|\mathbf{r}_T) - W_{\text{stat}}(\mathbf{r}_T)) - H(\mathbf{r}_T|\mathbf{r}_T) \right]. \tag{5}$$

We now show that $\lim_{T\to\infty} \mu(T) = 0$. Let ϵ be any fixed number. There exists T large enough such that:

$$| \left[\int_0^{T/2} dt (W(\mathbf{r}_T, t|\mathbf{r}_T) - W_{\text{stat}}(\mathbf{r}_T)) \right] - H(\mathbf{r}_T|\mathbf{r}_T)| \le \frac{\epsilon}{2} \tag{6}$$

and

$$\int_{T/2}^T dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \le \frac{\epsilon}{2\left[|H(\mathbf{r}_T|\mathbf{r}_T)| + \int_0^\infty dt |W(\mathbf{r}_T, t|\mathbf{r}_T) - W_{\text{stat}}(\mathbf{r}_T)| \right]}. \tag{7}$$

Then one has

$$\begin{aligned} |\mu(T)| \le \quad & | \int_0^{T/2} dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \left[\int_0^{T-t'} dt (W(\mathbf{r}_T, t|\mathbf{r}_T) - W_{\text{stat}}(\mathbf{r}_T)) - H(\mathbf{r}_T|\mathbf{r}_T) \right] | + \\ & + | \int_{T/2}^T dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \left[\int_0^{T-t'} dt (W(\mathbf{r}_T, t|\mathbf{r}_T) - W_{\text{stat}}(\mathbf{r}_T)) - H(\mathbf{r}_T|\mathbf{r}_T) \right] | \end{aligned} \tag{8}$$

In the first integral of (8), $T - t' \geq T/2$, so that

$$|\int_0^{T/2} dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \left[\int_0^{T-t'} dt(W(\mathbf{r}_T, t|\mathbf{r}_T) - W_{\text{stat}}(\mathbf{r}_T)) - H(\mathbf{r}_T|\mathbf{r}_T)\right] | \leq \frac{\epsilon}{2}\int_0^{T/2} dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \leq \frac{\epsilon}{2}, \tag{9}$$

and in the second integral

$$\begin{aligned} &|\int_{T/2}^{T} dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \left[\int_0^{T-t'} dt(W(\mathbf{r}_T, t|\mathbf{r}_T) - W_{\text{stat}}(\mathbf{r}_T)) - H(\mathbf{r}_T|\mathbf{r}_T)\right] | \\ \leq &\int_{T/2}^{T} dt' P(\mathbf{r}_T, t'|\mathbf{r}_S) \left[|H(\mathbf{r}_T|\mathbf{r}_T)| + \int_0^{\infty} dt |W(\mathbf{r}_T, t|\mathbf{r}_T) - W_{\text{stat}}(\mathbf{r}_T)|\right]] \leq \frac{\epsilon}{2}. \end{aligned} \tag{10}$$

Finally, $\mu(T) \leq \epsilon$, which proves that $\lim_{T\to\infty} \mu(T) = 0$.

Next remark that $T - \int_0^T TP(\mathbf{r}_T, t'|\mathbf{r}_S)dt'$ goes to 0 when T goes to infinity, since it is lower than $\int_T^\infty t' P(\mathbf{r}_T, t'|\mathbf{r}_S)dt'$, and the mean first-passage time is finite. We can thus write from the $T \to \infty$ limit of equation 4:

$$\langle \mathbf{T} \rangle = \frac{H(\mathbf{r}_T|\mathbf{r}_T) - H(\mathbf{r}_T|\mathbf{r}_S)}{W_{\text{stat}}(\mathbf{r}_T)}. \tag{11}$$

II. PSEUDO-GREEN FUNCTIONS

In this paragraph, we show that the function $H(\mathbf{r}|\mathbf{r}') \equiv \int_0^\infty (W(\mathbf{r}, t|\mathbf{r}') - W_{\text{stat}}(\mathbf{r}))dt$ defined above is the pseudo-Green function of the problem, namely that it satisfies

$$-\Delta_{\mathbf{r}} H(\mathbf{r}|\mathbf{r}') = -W_{\text{stat}}(\mathbf{r}) + \delta_{\mathbf{r},\mathbf{r}'}, \tag{12}$$

where $\Delta_{\mathbf{r}}$ is the Laplace operator of the walk defined by $\Delta_{\mathbf{r}} f(\mathbf{r}) = \sum_{\mathbf{r}''}[w_{\mathbf{r}\mathbf{r}''} f(\mathbf{r}'') - w_{\mathbf{r}''\mathbf{r}} f(\mathbf{r})]$, and $w_{\mathbf{r}'\mathbf{r}}$ stands for the transition probability from site $\mathbf{r}$ to site $\mathbf{r}'$. A complete introduction to pseudo-Green and Green functions can be found in ref.([1]). We use in this paper only the definition given by equation 12, and its connection with the infinite space Green function discussed in the next paragraph.

We start with the equations satisfied by the propagator W in *confined* space with reflecting boundary conditions :

$$\frac{\mathrm{d}}{\mathrm{dt}} W(\mathbf{r}, t|\mathbf{r}') = \Delta_{\mathbf{r}} W(\mathbf{r}, t|\mathbf{r}'), \tag{13}$$

and by the stationary probability distribution W_{stat}:

$$0 = \Delta_{\mathbf{r}} W_{\text{stat}}(\mathbf{r}). \tag{14}$$

Integrating the difference between these two equations over t between 0 and ∞ leads straightforwardly to equation 12, which shows that H is the so-called pseudo-Green function of the problem.

III. GREEN FUNCTIONS

A key step in our derivation is that the pseudo-Green function defined above can be approximated by the *infinite* space Green function G_0, which by definition[1] satisfies

$$-\Delta_{\mathbf{r}} G_0(\mathbf{r}|\mathbf{r}') = \delta_{\mathbf{r},\mathbf{r}'}, \tag{15}$$

Taking the $N \to \infty$ limit of equation 12 gives equation 15, which shows that our approximation is valid in the large system size limit. We denote by W_0 the infinite space propagator, which also satisfies equation 13. We next give useful integral representations of G_0 which depend on the relative order of d_f and d_w.

A. Case of non compact exploration : $d_w < d_f$

We write in this case

$$G_0(\mathbf{r}|\mathbf{r}') \equiv \int_0^\infty W_0(\mathbf{r},t|\mathbf{r}')dt. \tag{16}$$

Integrating over t between 0 and ∞ equation 13 written for W_0 leads to equation 15. G_0 is therefore the Green function of the problem.

B. Case of compact exploration : $d_w \geq d_f$

In this situation, we must treat separately the case of symmetric transition probabilities $w_{\mathbf{r}\mathbf{r}'} = w_{\mathbf{r}'\mathbf{r}}$, and the non symmetric case.

1. *Symmetric transition probabilities*

The integral 16 defining G_0 above is not finite for $d_w \geq d_f$. However, we can use the following generalized definition depending on an unimportant reference point $\mathbf{r}_2$:

$$G_0(\mathbf{r}|\mathbf{r}') - G_0(\mathbf{r}_2|\mathbf{r}') \equiv \int_0^\infty (W_0(\mathbf{r},t|\mathbf{r}') - W_0(\mathbf{r}_2,t|\mathbf{r}'))dt, \tag{17}$$

which is shown as previously to satisfy equation 15. Equation 17 therefore defines the Green function of the problem in this case.

2. General case

It is necessary to introduce an auxiliary function I, defined by:

$$I(\mathbf{r}_1|\mathbf{r}') - I(\mathbf{r}_2|\mathbf{r}') \equiv \int_0^\infty \left(\frac{W_0(\mathbf{r_1}|\mathbf{r}')}{X(\mathbf{r_1})} - \frac{W_0(\mathbf{r_2}|\mathbf{r}')}{X(\mathbf{r_2})} \right) dt, \tag{18}$$

where the "weights" $X(\mathbf{r})$ are proportional to the stationary probability when the domain is confined and satisfy

$$\sum_{\mathbf{r}'} w_{\mathbf{r}\mathbf{r}'} X(\mathbf{r}') = \sum_{\mathbf{r}'} w_{\mathbf{r}'\mathbf{r}} X(\mathbf{r}). \tag{19}$$

For example, we can take $X(\mathbf{r}) = 1$ for symmetrical transition probabilities, and $X(\mathbf{r}) = k(\mathbf{r})$ for small-world networks defined in the main text ($k(\mathbf{r})$ being the degree of the node $\mathbf{r}$). We now define

$$G_0(\mathbf{r}|\mathbf{r}') \equiv X(\mathbf{r}) I(\mathbf{r}|\mathbf{r}'). \tag{20}$$

Note that this expression defines G_0 up to a constant times the weight function. The following calculation shows that G_0 satisfies equation 15 and is therefore the Green function of the problem :

$$\begin{aligned}
-\Delta_{\mathbf{r}} G(\mathbf{r}|\mathbf{r}') &= \sum_{\mathbf{r}''} [w_{\mathbf{r}\mathbf{r}''} G_0(\mathbf{r}''|\mathbf{r}') - w_{\mathbf{r}''\mathbf{r}} G_0(\mathbf{r}|\mathbf{r}')] \\
&= -\sum_{\mathbf{r}''} w_{\mathbf{r}\mathbf{r}''} X(\mathbf{r}'') [I(\mathbf{r}''|\mathbf{r}') - I(\mathbf{r}|\mathbf{r}')] && (21) \\
&= -\sum_{\mathbf{r}''} w_{\mathbf{r}\mathbf{r}''} X(\mathbf{r}'') \int_0^\infty \left[\frac{W_0(\mathbf{r}''|\mathbf{r}')}{X(\mathbf{r}'')} - \frac{W_0(\mathbf{r}|\mathbf{r}')}{X(\mathbf{r})} \right] dt, && (22) \\
&= -\int_0^\infty \left[\sum_{\mathbf{r}''} w_{\mathbf{r}\mathbf{r}''} W_0(\mathbf{r}'', t|\mathbf{r}') - \sum_{\mathbf{r}''} w_{\mathbf{r}''\mathbf{r}} W_0(\mathbf{r}, t|\mathbf{r}') \right] dt && (23) \\
&= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}} W_0(\mathbf{r}, t|\mathbf{r}') dt && (24) \\
&= \delta_{\mathbf{r},\mathbf{r}'} && (25)
\end{aligned}$$

where Eq.(19) was used to get the lines (21) and (23).

IV. DERIVATION OF EQUATION (6) OF THE MAIN TEXT

We consider the case of a stationary distribution $W_{\text{stat}} = 1/N$. The pseudo Green function H is approximated by the infinite space Green function G_0 defined in the previous section. We now assume for $\mathbf{r} \neq \mathbf{r}'$ the standard scaling for W_0:

$$W_0(\mathbf{r}, t|\mathbf{r}') \sim t^{-d_f/d_w} \Pi \left(\frac{|\mathbf{r} - \mathbf{r}'|}{t^{1/d_w}} \right), \tag{26}$$

where the scaling function Π behaves smoothly for low ξ (long times), $\Pi(\xi) \sim C_0 - \xi^{\beta}$, and decays fast enough for high ξ (large distances). This assumption is very dimly restrictive since for most fractal models, including loopless fractals, critical percolation clusters and Sierpinski gaskets[2], the propagator is expected to behave like a stretched exponential, which would satisfy both hypotheses.

In the following, $W_0(|\mathbf{r}-\mathbf{r}'|, t)$ stands for $W_0(\mathbf{r}, t|\mathbf{r}')$. The computation of the MFPT depends on the relative order of d_f and d_w.

A. If $d_f > d_w$

One has

$$H(\mathbf{r}_T|\mathbf{r}_T) - H(\mathbf{r}_T|\mathbf{r}_S) \approx G_0(0) - G_0(r) \tag{27}$$

with

$$G_0(r) \approx r^{d_w-d_f}\int_0^{\infty}\frac{\Pi(u^{-1/d_w})}{u^{d_f/d_w}}du, \tag{28}$$

which leads to

$$\langle \mathbf{T}\rangle \sim N(A - Br^{d_w-d_f}). \tag{29}$$

Here A depends on the small scale properties of the walk:

$$A = \int_0^{\infty} W_0(0,t)\mathrm{d}t \tag{30}$$

and B reads:

$$B = \int_0^{\infty}\frac{\Pi(u^{-1/d_w})}{u^{d_f/d_w}}du. \tag{31}$$

B. If $d_f = d_w$

In this case only differences of Green functions are well defined, and we have to introduce an extra reference point R which will be unimportant in the final results.

$$H(\mathbf{r}_T|\mathbf{r}_T) - H(\mathbf{r}_T|\mathbf{r}_S) \approx \int_0^{\infty}(W_0(0,t) - W_0(r,t))dt = \int_0^{\infty}(W_0(0,t)) - W_0(R,t))dt + \int_0^{\infty}(W_0(R,t) - W_0(r,t))dt \tag{32}$$

The second difference in equation 32 reads:

$$
\begin{aligned}
G_0(R) - G_0(r) &\approx \lim_{T\to\infty}\left[\int_0^T \Pi\left(\frac{R}{t^{1/d_w}}\right)dt - \int_0^T \Pi\left(\frac{r}{t^{1/d_w}}\right)dt\right] \\
&= \lim_{T\to\infty}\int_{T/r^{d_w}}^{T/R^{d_w}} \frac{\Pi(u^{-1/d_w})}{u}du \\
&= -C_0 d_w(\ln(R) - \ln(r)).
\end{aligned} \tag{33}
$$

Given that the scaling relation 26 is accurate for large R, we take the limit $R \to \infty$ and obtain the scaling relation:

$$
\langle \mathbf{T} \rangle \sim N(A + B\ln r). \tag{34}
$$

with

$$
A = \lim_{R\to\infty}\left[\int_0^\infty (W_0(0,t)) - W_0(R,t))dt - B\ln R\right] \text{ and } B = C_0 d_w. \tag{35}
$$

C. If $d_f < d_w$

We write $\Pi(\xi) = C_0 - \Pi^*(\xi)$. Then, introducing as previously the reference point R, the second difference in equation 32 reads:

$$
\begin{aligned}
G_0(R) - G_0(r) &= -\int_0^{T/R^{d_w}} \frac{R^{d_w-d_f}}{u^{d_f/d_w}}\Pi^*(u^{-1/d_w})du + \int_0^{T/r^{d_w}} \frac{r^{d_w-d_f}}{u^{d_f/d_w}}\Pi^*(u^{-1/d_w})du \\
&= -R^{d_w-d_f}\int_0^\infty \frac{du}{u^{d_f/d_w}}\Pi^*(u^{-1/d_w}) + r^{d_w-d_f}\int_0^\infty \frac{du}{u^{d_f/d_w}}\Pi^*(u^{-1/d_w}),
\end{aligned} \tag{36}
$$

Taking again the reference point R to infinity, we obtain the sought scaling relation, provided that $d_f/d_w + \beta/d_w > 1$:

$$
\langle \mathbf{T} \rangle \sim N(A + Br^{d_w-d_f}), \tag{37}
$$

where

$$
A = \lim_{R\to\infty}\left[\int_0^\infty (W_0(0,t)) - W_0(R,t))dt - BR^{d_w-d_f}\right] \text{ and } B = \int_0^\infty \frac{du}{u^{d_f/d_w}}\Pi^*(u^{-1/d_w}). \tag{38}
$$

Note that if $d_f/d_w + \beta/d_w \leq 1$ the MFPT is infinite.

V. EFFECTIVE MEDIUM APPROXIMATION OF THE RANDOM BARRIER MODEL

We consider a distribution of transition rates

$$
\rho(\Gamma) = (\alpha/\Gamma)(\Gamma/\Gamma_0)^\alpha \tag{39}
$$

if $\Gamma < \Gamma_0$, and 0 otherwise. If we consider that the transition rates correspond to jumps above an energy barrier, this model corresponds to exponentially distributed energy barriers[3]. The scaling function $\Pi(\xi)$ for the random energy barriers model in dimension 2 is simply a Gaussian:

$$\Pi(\xi) = \frac{\exp(-\xi^2/(4D_{\text{eff}}))}{4\pi D_{\text{eff}}}. \tag{40}$$

The diffusion coefficient D_{eff} can be computed numerically[3] by solving the equation:

$$\int_0^\infty \rho(\Gamma)\frac{D_{\text{eff}} - \Gamma}{(z-2)D_{\text{eff}} + 2\Gamma} d\Gamma = 0, \tag{41}$$

where z is the coordination number, 4 for a square lattice.

The constant B is then given by equation 35, and we have:

$$\langle \mathbf{T} \rangle \sim N\left(A + \frac{1}{2\pi D_{\text{eff}}}\ln r\right). \tag{42}$$

VI. CONSTRUCTION OF FRACTAL SCALE FREE NETWORKS

A model of scale free networks bearing length scale invariant properties described in the main text has been defined recursively in ref. ([4,5]): the network grows by adding m new offspring nodes to each existing network node, resulting in well defined modules. In addition, modules are connected to each other through x random links (see figure (1)). Figure (2) gives a graphic representation of such a network.

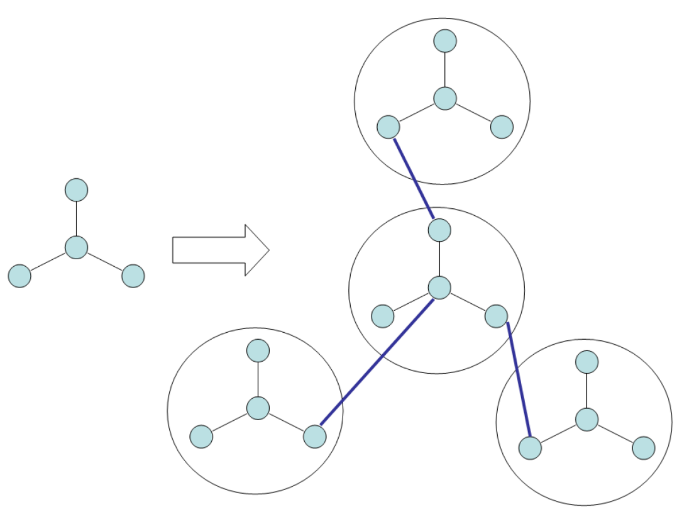

FIG. 1: A scale invariant scale free network defined in ref. ([5]), here with 2 generations of $m = 3$ offsprings each and $x = 1$.

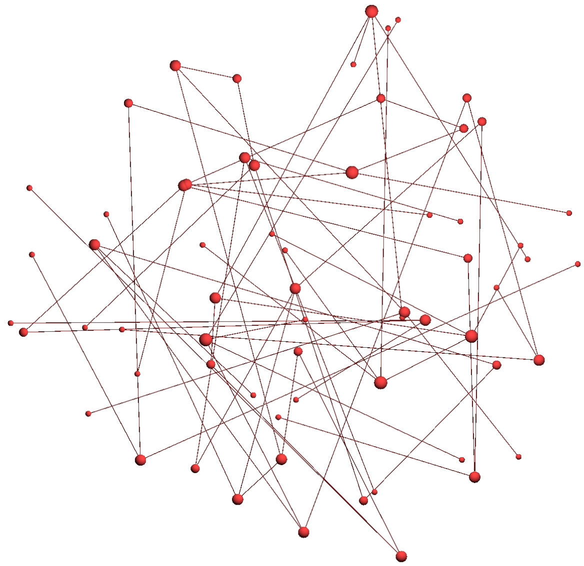

FIG. 2: A scale invariant scale free network defined in ref. ([5]), here with 4 generations of $m = 3$ offsprings each. The node diameter is proportional to the degree. Picture generated by the LaNet-vi software (http://xavier.informatics.indiana.edu/lanet-vi/)

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[2] S. Yuste, J.Phys.A **28**, 7027 (1995).

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[5] L. Gallos, C. Song, S. Havlin, and H. Makse, PNAS **104**, 7746 (2007).